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Minimal Generalized Interpolation Projections

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1. INTRODUCTION AND PRELIMINARIES

The problem of finding projections of minimal norm from $C[0, 1]$ onto the nth degree polynomial subspace $\mathscr{P}_n[0, 1]$ has been investigated by numerous authors, most notably Cheney, Morris, and Price (see, for example [2, 3, 4, 6, 7, 8]). The complete solution to this problem, even in the quadratic case $n = 2$, is unknown. It can be shown (see [1]), however, that among the minimal projections there is a *symmetric* projection P_S, satisfying $P_S[f(\cdot)](x) = P_S[f(1 - \cdot)](1 - x)$, for all $x \in [0, 1], f \in C[0, 1]$.

In this paper we investigate a subclass of these projections, called *generalized interpolating projections* (introduced in [4; Lemma 9]), and determine explicitly the minimal symmetric generalized interpolating projection in the quadratic case $n = 2$. Two representations for the minimum norm are provided below, and computational procedures based on each are illustrated for the quadratic case. Most of the theory in Sections 1 and 2 generalizes to arbitrary Haar subspaces of $C[0, 1]$.

Bounded projections from $C[0, 1]$ onto $\mathscr{P}_n[0, 1]$ can be represented in the form

$$P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i ,$$

where $\mathscr{L}_0 ,..., \mathscr{L}_n$ are independent bounded linear functionals on $C[0, 1]$, and $v_0 ,..., v_n \in \mathscr{P}_n[0, 1]$ are determined from $\mathscr{L}_i v_j = \delta_{ij}$ (Kronecker delta), i.e., the $\mathscr{L}_i$ and the v_j form a biorthogonal system.

DEFINITION. *A generalized interpolating projection* from $C[0, 1]$ onto $\mathscr{P}_n[0, 1]$ is a projection which has at least one representation $P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i$ in which the linear functionals $\mathscr{L}_0 ,..., \mathscr{L}_n$ have disjoint supports.

Notation. In the sequel, *generalized interpolating projection* will be abbreviated to gi projection or gip.

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The terminology gi projection is based on the interpolation property:

Given arbitrary constants $|c_i| < \|\mathscr{L}_i\| (i = 0,..., n)$, there exists an $f \in C[0, 1]$, $\|f\|_\infty = 1$, which interpolates c_i at $\mathscr{L}_i$ ($\mathscr{L}_i f = c_i$) for $i = 0,..., n$.

Then one also has that $Pf = \sum_{i=0}^n (\mathscr{L}_i f) v_i = \sum_{i=0}^n c_i v_i$ is an element of $\mathscr{P}_n$, in the range of the unit ball of $C[0, 1]$, which interpolates the values c_i at the $\mathscr{L}_i$. An immediate example of gi projections is given by the *interpolating projections*, i.e., those projections generated by taking $\mathscr{L}_i = e_{x_i}$, point evaluation at x_i, for $i = 0,..., n$.

Note. The projection $P = \sum_{i=0}^n \mathscr{L}_i \otimes v_i$ is invariant under an invertible linear transformation of the $(n + 1)$-tuple $\mathscr{L} = (\mathscr{L}_0 ,..., \mathscr{L}_n)$, since, for T a nonsingular $(n + 1) \times (n + 1)$ matrix, one has

$$P = \mathscr{L} \otimes v = (\mathscr{L}T) \otimes (vT^{t^{-1}}), \qquad v = (v_0 ,..., v_n),$$

where T^t denotes transpose. Thus $P = \mathscr{L} \otimes v$ is a gi projection if and only if there is an invertible transformation T such that the elements of $\mathscr{L}T$ are "disjoint" linear functionals.

From Cheney and Price [4; Lemma 9], one has the following fact.

PROPOSITION 1. *If* $P = \sum_{i=0}^n \mathscr{L}_i \otimes v_i$ *is a* gi *projection, normalized (without loss) so that all* $\|\mathscr{L}_i\| = 1$, *then*

$$\|P\| = \left\| \sum_{i=0}^n |v_i| \right\|_\infty .$$

A second representation for the norm of a gi projection is given in the following statement, which is a simple consequence of Proposition 1.

PROPOSITION 2. *If* $P = \sum_{i=0}^n \mathscr{L}_i \otimes v_i$ *is a* gi *projection, normalized (without loss) so that all* $\|\mathscr{L}_i\| = 1$, *then*

$$\|P\| = \sup \|p_n\|_\infty ,$$

where the supremum is taken over all $p_n \in \mathscr{P}_n$ *such that* $|\mathscr{L}_i p_n| = 1$ $(i = 0,..., n)$.

2. CHARACTERIZATIONS FOR THE INFIMUM OF THE NORMS OF GENERALIZED INTERPOLATION PROJECTIONS

In the previous section two characterizations were given for the norm of a gi projection. These will now be used to develop computationally useful characterizations for the infimum of the norms of gi projections (Theorems 2 and 3). A major additional tool in this development is the following known "quadrature-formula" type result (see, e.g., [5]). A short proof is given for the benefit of the reader.

THEOREM 1. *Let* $\hat{\mathscr{L}} \in \mathscr{P}_n^*[0, 1]$ *with* $\|\hat{\mathscr{L}}\| = 1$, *and suppose* $\mathscr{L}$ *is a norm* 1 *extension of* $\hat{\mathscr{L}}$ *to* $C^*[0, 1]$.

(a) *If* $\mathscr{L}$ *is signed, then* $\mathscr{L}$ *is supported on no more than* $n + 1$ *points (exactly* $n + 1$ *points implies the points are the Tchebycheff points on* $[0, 1]$).

(b) *If* $\mathscr{L}$ *is positive, then* $\mathscr{L}$ *can be replaced by another norm* 1 *extension of* $\hat{\mathscr{L}}$, $\mathscr{L}^*$, *which is positive and supported on no more than* $[(n + 2)/2]$ *points.*

Proof. (a) Suppose $\mathscr{L}$ is signed and has no fewer than $n + 1$ points in its support. Then $\|\mathscr{L}\| = \|\hat{\mathscr{L}}\| = 1$ implies that $\mathscr{L}$ achieves its norm at some $p \in \mathscr{P}_n$, where $\|p\| = 1$. But then $\mathscr{L}$ must have all its mass concentrated at the points x_i where $|p(x_i)| = 1$. Since there are at least $n + 1$ such points, p must be the nth degree Tchebycheff polynomial, and $\mathscr{L}$ has exactly the $n + 1$ "Tchebycheff points" as its support.

(b) Suppose $\mathscr{L}$ is positive and supported on more than n points. Then $\mathscr{L}$ gives rise to a positive measure μ on $C[0, 1]$, and induces an inner product $(f, g) = \int_0^1 fg\, d\mu$ on $C[0, 1]$. Let $r = [(n + 2)/2]$. Let x_i $(i = 1,..., r)$ be the roots of q_r, where $q_0,..., q_r$ are the orthogonal polynomials obtained from $1, x,..., x^r$ by the Gram-Schmidt orthogonalization process (with respect to the inner product $(\,,\,)$). Then the theory of orthogonal polynomials provides that there exist positive numbers a_i $(i = 1,..., r)$ such that $\mathscr{L}p = \sum_{i=1}^r a_i p(x_i)$ for all $p \in \mathscr{P}_n$. Taking $\mathscr{L}^* = \sum_{i=1}^r a_i e_{x_i}$, one has (i) $\mathscr{L} = \mathscr{L}^*$ on $\mathscr{P}_n$; and (ii) $\|\mathscr{L}^*\| = \sum_{i=1}^r a_i = 1$. ∎

Note. In the case of Theorem 1(a), $\mathscr{L}$ a signed functional, if $\mathscr{L}$ is supported on n points, then at least one of the endpoints must be included in the support of $\mathscr{L}$. This follows upon noting that an nth degree polynomial of norm 1 on $[0, 1]$ has at most $n - 1$ extrema in the open interval $(0, 1)$.

To distinguish the functionals described in Theorem 1, the terminology *simultaneously realizable* will be used.

DEFINITION. A linear functional $\mathscr{L} \in C^*[0, 1]$ is *simultaneously realizable* (*sr*) if it achieves its $C^*[0, 1]$ norm on the subspace $\mathscr{P}_n[0, 1]$. Further, if $\|\mathscr{L}\| = 1$, $\mathscr{L}$ will be said to be *normalized sr* (*nsr*).

EXAMPLE. Consider the quadratic case $n = 2$. Theorem 1(a) then yields a complete characterization of signed nsr functionals. In fact, such a signed functional must have one of the forms

(i) $\pm[\lambda e_0 - (1 - \lambda) e_x]$, $\quad \frac{1}{2} \leq x \leq 1, 0 \leq \lambda \leq 1$,

(ii) $\pm[\lambda e_1 - (1 - \lambda) e_x]$, $\quad 0 \leq x \leq \frac{1}{2}, 0 \leq \lambda \leq 1$,

(iii) $\pm[\alpha_1 e_0 - \alpha_2 e_{1/2} + \alpha_3 e_1]$, $\quad 0 \leq \alpha_i, \sum \alpha_i = 1$.

In case (i), the quadratic with value 1 at 0 and minimum value -1 at x yields $\pm \|\mathscr{L}'\|$ (noting that $\frac{1}{2} \leqslant x \leqslant 1$ is required to yield a quadratic of norm 1). Case (ii) is analogous. In case (iii), the quadratic with values 1 at 0 and 1, and value -1 at $\frac{1}{2}$ yields $\pm \|\mathscr{L}\|$.

Also in the case $n = 2$, Theorem 1(b) states that $\mathscr{L}^* = \lambda e_x + (1 - \lambda) e_y$ for some $0 \leqslant x \leqslant y \leqslant 1$, $0 \leqslant \lambda \leqslant 1$.

The following two theorems provide distinct characterizations for the infimum of the norms of gi projections. Each of these characterizations leads to a different numerical procedure for determining this infimum, as exemplified in Section 3, where the quadratic case $n = 2$ is discussed.

THEOREM 2. $\inf \|P^{\text{gip}}\| = \inf \|\sum_{i=0}^{n} |v_i|\|_{\infty}$, *where the second infimum ranges over all* $v_0, ..., v_n \in \mathscr{P}_n[0, 1]$ *adjusted so that the dual basis functionals* $\hat{\mathscr{L}}_0, ..., \hat{\mathscr{L}}_n \in \mathscr{P}_n^*[0, 1]$ (*i.e.*, $\hat{\mathscr{L}}_i v_j = \delta_{ij}$) *have norm* 1.

Proof. Given such $v_0, ..., v_n \in \mathscr{P}_n[0, 1]$, it will be shown that $\|\sum_{i=0}^{n} |v_i|\|_{\infty}$ is the limit of norms of gi projections. According to Theorem 1, extend each $\hat{\mathscr{L}}_i$ to $\mathscr{L}_i = \sum_{j=0}^{n} a_{ij} e_{x_{ij}}$, where $\|\hat{\mathscr{L}}_i\| = \sum_{j=0}^{n} |a_{ij}| = 1$. If there is overlap of support, let $x'_{ij} = x_{ij} + \epsilon_{ij}$ so that the supports of $\mathscr{L}_i' = \sum_{j=0}^{n} a_{ij} e_{x'_{ij}}$ are disjoint and in [0, 1]. Consider $v_0', ..., v_n' \in \mathscr{P}_n[0, 1]$, where $\mathscr{L}_i' v_j' = \delta_{ij}$. Then $P' = \sum_{i=0}^{n} \mathscr{L}_i' \otimes v_i'$ is a generalized interpolating projection. By Proposition 1.1, $\|P'\| = \|\sum_{i=0}^{n} |v_i'|\|_{\infty}$, which approaches $\|\sum_{i=0}^{n} |v_i|\|_{\infty}$ as $\epsilon_{ij} \to 0$ $(0 \leqslant i, j \leqslant n)$.

On the other hand, if $P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i$, where the $\mathscr{L}_i$ have disjoint supports ($\|\mathscr{L}_i\| = 1$), then $\|P\| = \|\sum_{i=0}^{n} |v_i|\|_{\infty}$ by Proposition 1. ∎

Remark. Note that, in the first part of the proof of Theorem 2, if $P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i$, then it is not in general true that $\|P'\| \to \|P\|$. Therefore, we cannot conclude that the infimum is attained. In the quadratic case, however, the infimum is attained (see Section 3).

THEOREM 3. $\inf \|P^{\text{gip}}\| = \inf \max_{p \in \mathscr{P}_n[0,1],\ |\mathscr{L}_i p| = 1} \|p\|_{\infty}$, *where the second infimum ranges over all linearly independent* nsr *functionals* $\mathscr{L}_0, ..., \mathscr{L}_n$.

Proof. It will first be shown that

$$\inf \|P^{\text{gip}}\| = \sup_{\epsilon} \inf \max_{\substack{p \in \mathscr{P}_n[0,1] \\ |\mathscr{L}_i p| = 1}} \|p\|_{\infty}, \qquad (\dagger)$$

where the second infimum is taken over all $\mathscr{L}_0, ..., \mathscr{L}_n$ which are disjoint functionals of norm 1 in $C^*[0, 1]$, and such that, if $\hat{\mathscr{L}}_i = \mathscr{L}_i|_{\mathscr{P}_n[0,1]}$, then $\|\hat{\mathscr{L}}_i\| \geqslant 1 - \epsilon$; call these functionals nsr (ϵ) functionals. The theorem is obtained by letting $\epsilon \to 0$, and observing that subsequential limits of nsr(ϵ) functionals are nsr functionals, and that on $\mathscr{P}_n[0, 1]$ the possible loss of disjointness in the limit has no effect on $\max \|p\|_{\infty}$.

To show (†), note that any gi projection written as $P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i$ (with $\|\mathscr{L}_i\| = 1$) has norm

$$\|P\| = \max_{\substack{p \in \mathscr{P}_n[0,1] \\ |\mathscr{L}_i p| \leq 1}} \|p\|_\infty = \left\| \sum_{i=0}^{n} |v_i| \right\|_\infty ,$$

the equalities following from Propositions 1.1 and 1.2, respectively. By the proof of Theorem 2 it follows that $\|\sum_{i=0}^{n} \|\hat{\mathscr{L}}_i\| \cdot |v_i|\|_\infty$ is a limit of norms of gi projections with nsr (ϵ) functionals. But

$$\left\| \sum_{i=0}^{n} \|\mathscr{L}_i\| \cdot |v_i| \right\|_\infty = \left\| \sum_{i=0}^{n} |v_i| \right\|_\infty = \|P\|. \quad ∎$$

3. Minimal Norm in the Quadratic Case $n = 2$

In this section, attention will be confined to the case $n = 2$ and, in particular, to the computation of the infimum of the norms of all symmetric gi projections by use of the two different procedures inherent in Theorems 2 and 3. As the computation will show, this infimum is actually attained by a gi projection.

To facilitate the computation, and to reduce the number of parameters involved, the following lemmas are needed.

Lemma 1. *Let $\hat{\mathscr{L}} \in \mathscr{P}_2^*[0,1]$ be represented by $\hat{\mathscr{L}} = a_1 e_0 + a_2 e_{1/2} + a_3 e_1$. Then*

(i) $\|\hat{\mathscr{L}}\| = \sum_{i=1}^{3} |a_i|$ *if* $a_1 a_3 \geq 0$ *or* $a_2 = 0$;

(ii) $\|\hat{\mathscr{L}}\| = a_1 + a_2 + a_3 - 2 \min\{0, a_3 + a_1 a_2 (4a_1 + a_2)^{-1}\}$ if $0 < a_1$, $0 < a_2$, $a_3 < 0$.

(The remaining cases may be determined from (ii) by symmetry.)

Proof. In case (i), $\|\hat{\mathscr{L}}\| = \sup \hat{\mathscr{L}} p_2$ over all p_2 in the unit ball of $\mathscr{P}_2[0,1]$, and is yielded by either $p_2 = \pm 1$, $p_2 = \pm(1 - 2x)$, or $p_2 = \pm[1 + 8x(x - 1)]$.

In case (ii), simple considerations show that $\hat{\mathscr{L}} p_2$ is largest if p_2 is concave downward and achieves its norm (1) as a maximum value. Let

$$p_{\rho,\theta}(x) = 1 - \rho(x - \theta)^2, \qquad 0 \leq \rho \leq 2/(1 - \theta)^2, \ 0 \leq \theta \leq \tfrac{1}{2},$$

represent an arbitrary quadratic having these characteristics. Then $\|\hat{\mathscr{L}}\| = \sup_{\rho,\theta} \hat{\mathscr{L}} p_{\rho,\theta}$, where

$$\hat{\mathscr{L}} p_{\rho,\theta} = a_1 + a_2 + a_3 - \rho[a_1 \theta^2 + a_2(\tfrac{1}{2} - \theta)^2 + a_3(1 - \theta)^2].$$

Hence, for each θ, $\hat{\mathscr{L}}p_{\rho,\theta}$ is linear in ρ, and the extremum is achieved for either $\rho = 0$ or $\rho = 2/(1-\theta)^2$. Thus one has

$$\|\hat{\mathscr{L}}\| = a_1 + a_2 + a_3 - 2\min\{0, \inf_\theta[a_1\theta^2 + a_2(\tfrac{1}{2}-\theta)^2 + a_3(1-\theta)^2] \times (1-\theta)^{-2}\}.$$

The result follows upon differentiation of the expression in θ, noting that the minimum occurs at $\theta = a_2/(4a_1 + 2a_2)$, and simplifying the resulting expression ∎

A symmetric projection can be written as $P = \sum_{i=1}^{3} \mathscr{L}_i \otimes v_i$, where $\mathscr{L}_1 f(\cdot) = \mathscr{L}_3 f(1-\cdot)$ and $\mathscr{L}_2 f(\cdot) = \mathscr{L}_2 f(1-\cdot)$. Hence, if $\mathscr{L}_i|_{\mathscr{P}_2} = \hat{\mathscr{L}}_i = a_{i1}e_0 + a_{i2}e_{1/2} + a_{i3}e_1$, then one has $a_{1j} = a_{3(4-j)}$ for $j = 1, 2, 3$, and $a_{21} = a_{23}$. Thus, for a symmetric gi projection, the triple $\hat{\mathscr{L}}_1, \hat{\mathscr{L}}_2, \hat{\mathscr{L}}_3$ (all of norm 1) can be represented by a matrix

$$\begin{pmatrix} a & b & c \\ d & 1-2|d| & d \\ c & b & a \end{pmatrix}, \qquad (*)$$

where $|d| \leq \frac{1}{2}$, and $\|\hat{\mathscr{L}}_1\| = 1$ determines c as a function of a and b (according to Lemma 1). Let $v_1, v_2, v_3 \in \mathscr{P}_2$ be dual to the functionals $\hat{\mathscr{L}}_1, \hat{\mathscr{L}}_2, \hat{\mathscr{L}}_3$ ($\hat{\mathscr{L}}_i v_j = \delta_{ij}$). From the symmetry of the $\hat{\mathscr{L}}_i$, one has $v_1(x) = v_3(1-x)$ and $v_2(x) = v_2(1-x)$ for $0 \leq x \leq 1$.

Lemma A will provide the gross estimates which facilitate the proofs of Lemmas 2, 3, and 4. Recalling that the interpolating projection P_I carried on $\{0, \frac{1}{2}, 1\}$ has norm 5/4, we now prove the following "continuity" result. Simple considerations of symmetry show that one can assume $a \geq |c|$.

LEMMA A. *If P is a symmetric* gip *onto $\mathscr{P}_2$ with norm $\leq 5/4$, then P is "close to" P_I in the sense that, in $(*)$, $a \geq 4/5$, $|c| \leq 1/5$, $|b| \leq 2/5$, $|d| \leq 1/10$. Also, $a + \frac{1}{2}|b| + |c| \leq 1$.*

Proof. If $a - c < 4/5$, then consider $p \in \mathscr{P}_2$ satisfying $p(\frac{1}{2}) = 0$, $p(1) = 1/(a-c)$ and $p(0) = 1/(c-a)$. Then $\mathscr{L}_1 p = 1$, $\mathscr{L}_2 p = 0$. $\mathscr{L}_3 p = -1$; and hence $\|P\| \geq \|p\|_\infty = 1/(a-c) > 5/4$. This contradiction gives $a - c \geq 4/5$. A simple calculation using Lemma 1 then shows that $|b| \leq 2/5$.

We now show that $|d| \leq 1/10$, i.e., $1 - 2|d| \geq 4/5$. If $1 - 2|d| < 4/5$, then consider $p \in \mathscr{P}_2$ satisfying $p(0) = p(1) = 0$ and $p(\frac{1}{2}) = \min(5/2, 1/(1-2|d|))$. But $\mathscr{L}_1 p = \mathscr{L}_3 p = bp(\frac{1}{2})$ while $\mathscr{L}_2 p = (1/(1-2|d|))\, p(\frac{1}{2})$. Thus $\|P\| \geq \|p\|_\infty > 5/4$, since $|\mathscr{L}_i p| \leq 1$, $i = 1, 2, 3$.

Consider now $p \in \mathscr{P}_2$ satisfying $p(0) = 1/a$, $p(\frac{1}{2}) = p(1) = 0$. Then $\mathscr{L}_1 p = 1$, $\mathscr{L}_3 p = c/a$, $\mathscr{L}_2 p = d/a$. Now if $a < 1/10$, then $|c| < 1/10$ and since $|b| < 2/5$, this contradicts $\|\hat{\mathscr{L}}\| = 1$. Thus $a \geq 1/10$; and since $|d| \leq 1/10$, we have $|\mathscr{L}_2 p| \leq 1$. Recall that $|c| \leq a$; hence also $|\mathscr{L}_3 p| \leq 1$. We conclude that

$a \geq 4/5$, since otherwise $\|P\| \geq \|p\|_\infty > 5/4$. But also since $\|\hat{\mathscr{L}}_1\| = 1$, we have that $a + |c| \leq 1$, and hence $|c| \leq \frac{1}{5}$.

Finally the conclusion that $a + \frac{1}{2}|b| + |c| = 1$ follows from $\hat{\mathscr{L}}_1 p = 1$ for all $p \in \mathscr{P}_2$ of norm 1. ▌

LEMMA 2. *If* $\inf \|p^{\text{gip}}_{\text{symm}}\| = \|\sum_{i=1}^{3} |v_i|\|_\infty$, *then* $\mathscr{L}_2 = e_{1/2}$ $(d = 0)$ *and* $0 \leq b$ *in* $(*)$.

Proof. Suppose $\|\sum_{i=1}^{3} |v_i|\|_\infty = \pm v_1(x_1) \pm v_2(x_1) \pm v_3(x_1)$ for some $x_1 \in [0, 1]$ and a particular choice of the $\pm$ signs; and consider $p = \pm v_1 \pm v_2 \pm v_3 \in \mathscr{P}_2$. Since $v_1 - v_2 - v_3$ is symmetrical to $v_1 + v_2 - v_3$, and since we will show at the end of the proof that $v_1 + v_2 + v_3$ is dominated by the $v_1 - v_2 + v_3$ case, there are only two relevant choices for p; namely, the symmetrical (about $\frac{1}{2}$) case $p^{(1)} = v_1 - v_2 + v_3$ corresponding to p taking values $(1, -1, 1)$ at $(\hat{\mathscr{L}}_1, \hat{\mathscr{L}}_2, \hat{\mathscr{L}}_3)$, and the nonsymmetrical case $p^{(2)} = v_1 + v_2 - v_3$ corresponding to p taking values $(1, 1, -1)$ at $(\hat{\mathscr{L}}_1, \hat{\mathscr{L}}_2, \hat{\mathscr{L}}_3)$.

Lemma A shows that $p^{(1)}$ and $p^{(2)}$ have the pictorial representations indicated below in Fig. 1. We will use this picture only for illustrative purposes.

Since $v_3(x) = v_1(1 - x)$, one has $p^{(2)}(x) = v_1(x) - v_1(1 - x) + v_2(x)$, where

$$v_1(x) - v_1(1 - x) = \alpha(x - \tfrac{1}{2}) \qquad \text{and} \qquad v_2(x) = -\beta(x - \tfrac{1}{2})^2 + \gamma$$

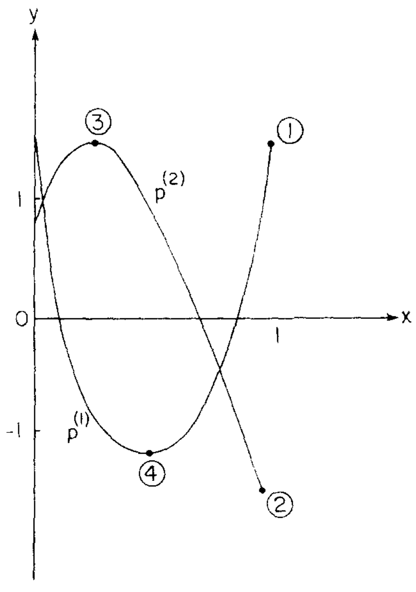

FIGURE 1.

for some α, β, γ. The determination of α, β, γ may be accomplished by using the relations $\hat{\mathscr{L}}_1 p^{(2)} = 1$, $\hat{\mathscr{L}}_2 p^{(2)} = 1$, and $\hat{\mathscr{L}}_1 v_2 = 0$; the result being $\alpha = -2/(a-c)$, $\beta = 2r\gamma$, $\gamma = 1/(1-dr)$ (if $d \geqslant 0$), $\gamma = 1/(1+(4-r)d)$ (if $d < 0$), where $r = 2(a+b+c)(a-c)^{-1}$. Thus,

$$p^{(2)}(x) = \alpha(x-\tfrac{1}{2}) + \gamma[1-2r(x-\tfrac{1}{2})^2],$$

where γ is the *only* quantity depending on the parameter d.

Lemma A yields the gross estimates $|b| < a+c$, $|r-2| < 1$, and $|d| < 1/10$. The maximum of $p^{(2)}$, point ③ in Fig. 1, is then minimized by taking $d = 0$, i.e., $\mathscr{L}_2 = e_{1/2}$. Also, point ② has depth

$$p^{(2)}(1) = (\alpha/2) + \gamma((r/2) - 1),$$

where $\alpha < 0$ and $2 \leqslant r$. Hence, this depth is minimized by taking γ as small as possible, i.e., $d = 0$,

We show now that we can assume $b \geqslant 0$. Note that $p^{(2)}(\tfrac{1}{4}) = -\alpha/4 + \gamma(1-r/8)$. Since $\gamma \geqslant 1$, if $b < 0$, then $r < 2$ and a simple calculation shows that $a-c = 1 + b(2c+b)/(4c+b)$ yielding $a-c < 1$, and so $-\alpha > 2$. We conclude that $p^{(2)}(\tfrac{1}{4}) > 5/4$.

In the case of $p^{(1)}$, one has $p^{(1)}(0) = p^{(1)}(1)$; so that

$$\mathscr{L}_1 p^{(1)} = 1 \text{ yields } (a+c)\,p^{(1)}(1) + bp^{(1)}(\tfrac{1}{2}) = 1, \qquad (*_1)$$

$$\mathscr{L}_2 p^{(1)} = -1 \text{ yields } \begin{cases} 2dp^{(1)}(1) + (1-2|d|)\,p^{(1)}(\tfrac{1}{2}) = -1 & (\text{if } d < 0), \\ \tfrac{1}{2}[p^{(1)}(x_0) + p^{(1)}(1-x_0)] = -1 & (\text{if } d \geqslant 0), \end{cases} \qquad (*_2)$$

(see Theorem 2.1b). Suppose $d \geqslant 0$; the second condition then shows that, in order to minimize the depth of point ④, one should take $x_0 = \tfrac{1}{2}$ (i.e., $d = 0$). The first condition shows that, as $p^{(1)}(\tfrac{1}{2})$ is raised (letting $x_0 \to \tfrac{1}{2}$), the height of point ① is not increased. Next, suppose $d < 0$. Letting $\rho = (a+c)(1-2|d|) - 2db$, we have $p^{(1)}(1) = (1+b-2|d|)\,\rho^{-1}$ and $p^{(1)}(\tfrac{1}{2}) = -(a+c+2d)\,\rho^{-1}$. A simple calculation shows that $p^{(1)}(1)$ is minimal if $d = 0$. A further simple calculation (involving several cases) shows that $|p^{(1)}(\tfrac{1}{2})| \leqslant p^{(1)}(1)$ for all d.

We now show that $v_1 + v_2 + v_3$ is dominated by $v_1 - v_2 + v_3$. In this case we obtain relations identical to $(*_1)$ and $(*_2)$ except that in $(*_2)$ $+1$'s replace -1's on the right-hand sides of the equalities. Using $\mathscr{L}_2 = e_{1/2}$, we have that $p^{(3)}(1) = (1-b)/(a+c)$ and $p^{(3)}(\tfrac{1}{2}) = 1$. But $p^{(3)}(1) = (1-b)/(a+c) < (1+b)/(a+c) = p^{(1)}(1)$. ∎

Note. An analysis of the whole situation in the gip quadratic case reveals that the essential factor for determining the minimal projection is a Chebyshev-type balancing between points ③ (interior maximum) and ① (endpoint maximum) in Fig. 1; and, in fact, points (2) and (4) in the same figure are not critical.

DEFINITION. If P can be written $P = \sum_{i=0}^{n} \mathscr{L}_i \otimes v_i$, where all $\mathscr{L}_i$ are positive functionals and disjoint, then P will be called a *positively representable* gi *projection*.

LEMMA 3. *The interpolating projection at* $0, \frac{1}{2}, 1$ *is the minimal projection among all positively representable symmetric* gi *projections onto the quadratics.*

Proof. The proof of Lemma 2 shows that the statement of Lemma 2 also holds if the infimum is taken over merely the positively representable symmetric gi projections. Thus, one need consider only the situation.

$$\mathscr{L}_1 = \lambda e_0 + (1-\lambda)\, e_z, \quad \mathscr{L}_2 = e_{1/2}, \quad \mathscr{L}_3 = \lambda e_1 + (1-\lambda)\, e_{1-z},$$

where $0 < \lambda < 1$ and $z \neq 0, \frac{1}{2}$.

Consider then the case where p takes the values 1, 1, -1 at $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$. As in Lemma 2, the conditions $\mathscr{L}_1 p = 1$, $\mathscr{L}_2 p = 1$, $\mathscr{L}_3 p = -1$ may be used to determine explicitly p in terms of λ and z. The interior maximum may then be computed to be

$$\max = 1 + \tfrac{1}{4}(r/s^2), \qquad \text{where} \quad \begin{aligned} r &= (\lambda/4) + (1-\lambda)(z-\tfrac{1}{2})^2, \\ s &= (1-\lambda)(z-\tfrac{1}{2}) - (\lambda/2). \end{aligned}$$

Differentiating $1 + 4r/s^2$ with respect to z yields

$$d\max/dz = -\tfrac{1}{4}\,(\lambda(1-\lambda)\, z/s^3),$$

which is positive since $s = -[\lambda(\frac{1}{2}) + (1-\lambda)(\frac{1}{2} - z)] < 0$. Hence, even the interior extremum decreases to 5/4 as $z \to 0$. ∎

LEMMA 4. *The interpolating projection at* $0, \frac{1}{2}, 1$ *is the minimal projection among all symmetric* gi *projections where* $\mathscr{L}_1$ *is signed and* $\hat{\mathscr{L}}_1$ *has the form* $\lambda e_0 - \mu e_{1/2} + (1-\lambda-\mu)\, e_1$, *with* $0 < \lambda, \mu, 1-\lambda-\mu$ *and* $2\lambda + \mu \neq 1$ (*independence of* $\hat{\mathscr{L}}_1$, $\hat{\mathscr{L}}_2$, $\hat{\mathscr{L}}_3$).

Proof. The proof of Lemma 2 shows that the statement of Lemma 2 also holds if the infimum is taken over all symmetric gi projections having $\mathscr{L}_1$ as described. Thus, one need consider only the situation

$$\begin{aligned} \hat{\mathscr{L}}_1 &= \lambda e_0 - \mu e_{1/2} + (1-\lambda-\mu)\, e_1, \\ \mathscr{L}_2 &= e_{1/2}, \\ \mathscr{L}_3 &= (1-\lambda-\mu)\, e_0 - \mu e_{1/2} + \lambda e_1. \end{aligned}$$

As in Lemma 3, it will suffice to consider the case where p takes the values $1, 1, -1$ at $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$. Using these conditions, p may be computed as

$$p(x) = 1 + \alpha(x - \tfrac{1}{2}) - \beta(x - \tfrac{1}{2})^2,$$

$$\alpha = \frac{2}{1 - 2\lambda - \mu}, \qquad \beta = \frac{4(1 - 2\mu)}{1 - \mu}.$$

If $\mu = \frac{1}{2}$, p is linear with $\| p \|_\infty$ achieved at either 0 or 1. But, in this case,

$$p(0) = 1 - (\alpha/2), \qquad p(1) = 1 + (\alpha/2);$$

which gives

$$\| p \|_\infty = 1 + (|\alpha|/2) = 1 + (1/|1 - 2\lambda - \mu|) \geqslant 2.$$

If $\mu \neq \frac{1}{2}$, the extreme values of p are given by

$$1 \pm \frac{\alpha}{2} - \frac{\beta}{4} = \frac{\mu}{1 - \mu} \pm \frac{1}{1 - 2\lambda - \mu} \qquad \text{at} \quad x = 0, 1;$$

$$1 + \frac{\alpha^2}{4\beta} = 1 + \frac{1 - \mu}{4(1 - 2\mu)(1 - 2\lambda - \mu)^2} \qquad \text{at} \quad x_{\text{int}} = \frac{1}{2} + \frac{\alpha}{2\beta}$$

$$(\text{if } x_{\text{int}} \in [0, 1]).$$

A close analysis of these extrema shoms that $\| p \|_\infty \geqslant 5/4$. ∎

Numerical procedure I. Theorems 2.1 and 2.2 provide that the infimum of the norms of all symmetric gi projections onto the quadratics may be obtained as follows. Invert the matrix

$$\begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ c & b & a \end{pmatrix}, \qquad \begin{array}{l} 4/5 \leqslant a \leqslant 1 \\ 0 \leqslant b \leqslant \tfrac{1}{2}[1 - 3a + (1 + 10a - 7a^2)^{1/2}] \\ c = a + b - 2ab(4a + b)^{-1} - 1 \end{array} \qquad (**)$$

to obtain

$$(v_{ij}) = (1/(a^2 - c^2)) \begin{pmatrix} a & -b(a - c) & -c \\ 0 & a^2 - c^2 & 0 \\ -c & -b(a - c) & a \end{pmatrix}.$$

(Lemmas 1, 2, 3, 4 guarantee that the restrictions in $(**)$ are admissible.) The columns of this last matrix provide the values of v_j at $0, \frac{1}{2}, 1$, i.e.,

$$v_j((i - 1)/2) = v_{ij}, \qquad i = 1, 2, 3 \text{ and } j = 1, 2, 3.$$

If

$$N(a, b) = \Big\| \sum_{j=1}^{3} |v_j| \Big\|_\infty = \max_{\pm \text{choices}} \| v_1 \pm v_2 \pm v_3 \|_\infty ,$$

then

$$\inf \| P^{\text{gip}}_{\text{symm.}} \| = \inf_{a,b} N(a, b).$$

This last infimum was determined by means of a two-parameter (a and b) search technique on a Hewlett–Packard 9830A programmable calculator. For the results, see Theorem 1 below.

Numerical procedure II. Theorems 2.1 and 2.3 provide that the infimum of the norms of all symmetric gi projections onto the quadratics may be obtained as follows. For $0 < \lambda < 1$ and $0 < z < \frac{1}{2}$, consider the functionals

$$\mathscr{L}_1 = \lambda e_z + (1 - \lambda) e_1 , \qquad \mathscr{L}_2 = e_{1/2} , \qquad \mathscr{L}_3 = \lambda e_{1-z} + (1 - \lambda) e_0 .$$

(Lemmas 1, 2, 3, 4 guarantee that the indicated restrictions on $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ are admissible.) If

$$N(\lambda, z) = \max_{|\mathscr{L}_i p| \le 1} \| p \|_\infty = \max_{\pm \text{choices}} \| p : \mathscr{L}_i p = \pm 1 \|_\infty ,$$

then

$$\inf \| P^{\text{gip}}_{\text{symm.}} \| = \inf_{\lambda, z} N(\lambda, z).$$

Again, a two-parameter search technique was used to obtain the results in the following theorem.

THEOREM 1. *In the quadratic case,*

$$\inf \| P^{\text{gip}}_{\text{symm.}} \| = 1.24839.$$

The infimum is uniquely (see the following note) achieved for $P = \sum_{i=1}^{3} \mathscr{L}_i \otimes v_i$, where

$$\mathscr{L}_1 = 0.94876\, e_{0.014322} + 0.05124\, e_1 ,$$
$$\mathscr{L}_2 = e_{1/2} ,$$
$$\mathscr{L}_3 = 0.94876\, e_{0.985678} + 0.05124\, e_0 .$$

Note. Uniqueness is obtained in the following sense. For each fixed z, a convex function of λ is being minimized. The resulting function of the single variable z shows, to the accuracy of the computation, a unique minimum.

Remark. While the above numerical procedures were of equal difficulty, the situation would appear to be different for $n > 2$. Procedure I required the calculation of the norm (on $\mathscr{P}_2$) of each functional $\mathscr{L}_i$. Even in the cubic case this calculation appears difficult, and would seem to indicate that Procedure II is preferable.

Remark. The restriction of considering only symmetric gi projections has some basis in numerical experiments. Numerous random searches were conducted without using the assumption of symmetry; the result being, in each case, an indication that symmetry yielded lower norms. At this time, the authors have been unable to establish, in a rigorous way, that symmetry must hold amongst a subset of the minimal gi projections (as is the case for the set of minimal projections: see [1]).

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